§4. Compactification on general 3-manifolds

§4.1 A-model labeled by 3-manifolds: $T^A[M_3]$

A general 3d $\mathcal{N}=2$ admits partial top. twist on $S^1 \times \Sigma$:

$$SO(2)'_\Sigma \subset SO(2)_\Sigma \times U(1)_t$$

where $U(1)_t$ is R-sym. of 3d $\mathcal{N}=2$ th.

$$\Big\downarrow S^1$$

2d $\mathcal{N}=(2,2)$ th on Σ with A-twist

Review of $\mathcal{N}=(2,2)$ in 2d:

- right-moving supercharges: Q_+, $\overline{Q}_+$
- left-moving supercharges: Q_-, $\overline{Q}_-$
- Hamiltonian: $H_\pm = (H \pm P)/2$
- $Q_r^2 = H_+$, where $Q_r = (Q_\pm + \overline{Q}_\pm)/2$
- elliptic genus: $\mathrm{Tr}\, q^{H} e^{i\gamma J_L} e^{i\alpha J_r}$

- two top. twists:

  A-twist: Q_-, $\bar{Q}_+$ have zero spin

  B-twist: $\bar{Q}_-$, $\bar{Q}_+$ have zero spin

$\to$ BRST-op:
$$Q = \begin{cases} Q_- + \bar{Q}_+ & (\text{A-twist}) \\ \bar{Q}_- + \bar{Q}_+ & (\text{B-twist}) \end{cases}$$

$\to$ elements of Q-cohomology form a ring $\mathcal{R}$:

A-twist: (a, c) - ring

anti-chiral chiral

$$\mathcal{R} = H^*(X) \quad , \quad (Q, Q^\dagger, H) \sim (d, d^*, \Delta)$$

target space of sigma model

One of the basic ingredients in $\mathcal{N}=(2,2)$ 2d theories is a free chiral superfield
$$\Phi = \phi + \theta^+\psi_+ + \theta^-\psi_- + \cdots$$

	$U(1)_\Sigma$	$U(1)_V$	$U(1)_A$	$m=0$
ϕ	0	m	n	z
ψ_-	1	$m-1$	$n+1$	dz
$\bar{\psi}_+$	-1	$m+1$	$n+1$	$d\bar{z}$
$\bar{\psi}_-$	1	$m+1$	$n-1$	
ψ_+	-1	$m-1$	$n-1$	

A-twist replaces $U(1)_\Sigma$-charge by sum of $U(1)_\Sigma$- and $U(1)_V$-charges

$\xrightarrow{m=0}$ $\phi, \psi_-, \bar\psi_+$ become scalars

$\rightarrow$ observables $\mathcal{O} \in H^{p,q}(X)$ correspond to R-charges $(m,n) = (q-p, q+p)$

$U(1)_V$ $(= U(1)_t)$ allows to define "graded" Q-cohomology:

$$\mathcal{H}(\Sigma) = \mathbb{Z}\text{-graded } Q\text{-cohomology}$$

$$\chi(\mathcal{H}(\Sigma)) = Z_{A\text{-model}}(\Sigma)$$

refined A-model partition function:

$$\dim_t \mathcal{H}(\Sigma) = \sum_j t^j \dim \mathcal{H}^j(\Sigma)$$

The top. A-model is 2d TQFT

$\rightarrow$ described by Frobenius algebra:

$$\eta_{ij} = \langle \phi_i \phi_j \rangle_0 = \langle i | j \rangle, \quad \phi_i | j \rangle = C_{ij}^{k} | k \rangle$$

$$\eta^{ij} \eta_{jk} = \delta^i_k, \quad C_{ijk} = \langle \phi_i \phi_j \phi_k \rangle_0 = \langle i | \phi_j | k \rangle$$

$$= \eta_{il} C^l_{jk}, \quad \phi_i \in R$$

$\rightarrow$ partition function can be computed on any Riemann surface Σ by surgery:

$$\langle \phi_{a_1} \cdots \phi_{a_n} \rangle_g = \sum_{ij} \langle \phi_{a_1} \cdots \phi_{a_r} \phi_i \rangle_{h} \eta^{ij} \langle \phi_j \phi_{a_{r+1}} \cdots \phi_{a_n} \rangle_{g-h}$$

Inserted operators lift to line operators in 3d:

$$\langle \phi_1 \cdots \phi_n \rangle_g = \mathrm{Tr}_{\mathcal{H}(\Sigma_g, \phi_1 \ldots \phi_n)} (-1)^F$$

Hilbert space of 3d th with line operators supported on $\phi_1, \ldots, \phi_n$

Now let us focus on particular 3d $\mathcal{N}=2$ theory $T_G[M_3] \xrightarrow{\text{A-model}} T_G^A[M_3]$

simple example: lens space theory

$T^A_{U(1)}[L(p,1)]$: 2d $\mathcal{N}=(2,2)$ twisted chiral with $\widetilde{W} = p\sigma^2/2$ and a free chiral

twisted superpotential $\widetilde{W}$ has $(m,n)=(0,2)$ charge under $U(1)_V \times U(1)_A$

The (a,c)-ring is given by

$$\mathcal{R} = \mathbb{C}[z]/(z^p - 1) \cong \mathbb{C}[\mathbb{Z}_p] \qquad (*)$$

where $z = e^{\sigma}$

and the condition $z^p - 1 = 0$ arises from:

$$\exp\left(\frac{\partial \widetilde{W}}{\partial \sigma}\right) = 1 \qquad \text{(vacua of 2d theory)}$$

The elements $(*)$ are lifted to anyonic Wilson lines in 3d $U(1)_p$ bosonic CS-theory and the multiplication in $\mathcal{R}$ agrees with the fusion rules

<u>General M_3:</u> reduction of the 6d $(2,0)$-th can then be viewed as:

6d $(2,0)$-th on $\Sigma \times S^1 \times M_3$

2d A-model on Σ with target $\mathcal{M}_{flat}(G_{\mathbb{C}}, M_3)$ $=$ 4d DW top. twist of $T_G[\Sigma]$ on $S^1 \times M_3$

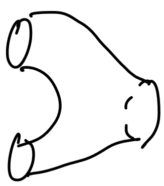
We have

$$\mathcal{R} \cong \mathcal{H}_{T^A[M_3]_G}(S^1) \cong H^*(\mathcal{M}_{flat}(G_{\mathbb{C}}, M_3))$$

In many examples, $\mathcal{M}_{flat}(G_{\mathbb{C}}, M_3)$ will be simply a discrete set (i.e. for lens spaces). In such a case,

$$\mathcal{H}_{T^A[M_3]}(S^1) = \mathbb{C}[\mathcal{M}_{flat}(G_{\mathbb{C}}, M_3)]$$

equipped with a chiral ring structure:



$$: \mathbb{C}[\mathcal{M}_{flat}(G_{\mathbb{C}}, M_3)] \times \mathbb{C}[\mathcal{M}_{flat}(G_{\mathbb{C}}, M_3)] \xrightarrow{m} \mathbb{C}[\mathcal{M}_{flat}(G_{\mathbb{C}}, M_3)]$$

$SL(2,\mathbb{Z})$ action:

We have

$$\mathcal{H}_{T^A[M_3]}(S^1) = \mathcal{H}_{T[M_3]}(T^2) \quad (1)$$

Since T^2 has mapping class group $SL(2,\mathbb{Z})$, it follows that (1) should be a (projective) representation of $SL(2,\mathbb{Z})$:

$$\mathcal{R} : SL(2,\mathbb{Z}) \to \mathrm{End}(\mathcal{H}_{T^A[M_3]}(S^1)) \quad (2)$$

First, let us assume that the fundamental group $\pi_1(M_3)$ is finite and

$$\mathcal{M}_{flat}(G_{\mathbb{C}}, M_3) = \mathcal{M}_{flat}(G, M_3)$$

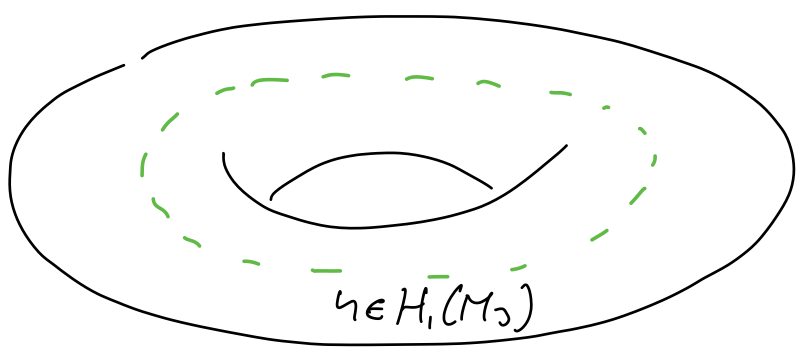
Consider the case $G = U(1)$ and set $H = H_1(M_3)$

$$\rightarrow \mathcal{M}_{flat}(U(1), M_3) \cong \mathrm{Hom}(H, U(1)) =: \hat{H}$$

where $\hat{H}$ is the Pontryagin dual of H

$$\rightarrow \mathcal{H}_{T^A[M_3]}(S^1) = \mathbb{C}[\hat{H}] \cong \mathbb{C}[H]$$

Introducing a line operator labelled by $h \in H$ along a non-trivial cycle of the solid torus creates a state in $\mathcal{H}_{T^A[M_3]}(S^1) = \mathbb{C}[H]$:



Let us now construct the representation of $SL(2,\mathbb{Z})$:

Consider any 4-manifold M_4 such that

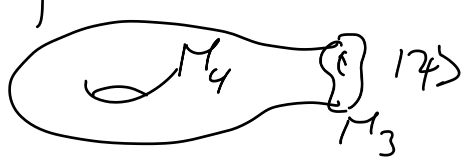
$$\partial M_4 = M_3$$

with $M_4 \sim M_3 \times \mathbb{R}$ near the boundary

Consider 6d (2,0)-th on $M_4 \times T^2$ with top. twist (to be explained later)

$\longrightarrow$ "Vafa-Witten"-TQFT in 4d

top. twisted $\mathcal{N}=4$ SYM



$$\longrightarrow \quad Z_{VW}(M_4)(\underset{\substack{\uparrow \\ \text{modular} \\ \text{parameter of torus}}}{\tau}) \in \mathcal{H}_{VW}(M_3)$$

and $\mathcal{H}_{VW}(M_3) = \mathcal{H}_{T[M_3 \times T^2]} = \mathcal{H}_{T[M_3]}(T^2)$

$SL(2,\mathbb{Z})$-action:

$$Z_{VW}(M_4)\left(\frac{a\tau+b}{c\tau+d}\right)$$

$$= C(\tau,a,b,c,d)\,\mathcal{R}\begin{bmatrix} a & b \\ c & d \end{bmatrix} Z_{VW}(M_4)(\tau)$$

The natural boundary condition for VW theory on M_4 requires gauge con. to approach a flat one at boundary

$$\longrightarrow \quad Z_{VW}(M_4)_\rho(\tau), \quad \rho \in \widehat{M}_{flat}(G, M_3)$$

Moreover, we have, defining $q = e^{2\pi i \tau}$,

$$Z_{VW}(M_4)_\rho(\tau) = \sum_n q^n \chi(\mathcal{M}^{inst}_{n,\rho})$$

where n is the instanton number